Marked Multitarget Intensity Filters

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Abstract – *Probability Hypothesis Density and other intensity filters are based on modeling the multitarget state as a realization of a Poisson point process (PPP). Target identifiability is lost in these models; consequently, the filters require targets to have the same motion models and data likelihood functions to be the same for all targets. These are unrealistic limitations in some applications.*

The Marked Multitarget Intensity Filter (MMIF) presented here enables the use of heterogeneous target motion models and data likelihood functions. The MMIF uses a marked PPP target model together with a parameterized PPP intensity function. The parametric model is an affine, joint, linear-Gaussian sum on the joint measurement-target space. The "at most one measurement per target" rule is enforced in the mean.

Keywords: Intensity filter, PHD filter, Heterogeneous target models, Poisson point process, Microtargets, Marked process, Expectation-Maximization, Gaussian sum filter, Marking Theorem.

1 Introduction

The Probability Hypothesis Density (PHD) and other intensity filters assume that targets have the same motion model. They also assume that the likelihood functions of the data are the same for all targets. Such assumptions are idealized and unrealistic in some applications.

An alternative approach to these Poisson point process (PPP) based filters called the Marked Multitarget Intensity Filter (MMIF) is presented. The MMIF accommodates heterogeneous target motion models and measurement pdfs (probability density functions). The added generality is achieved by using a marked PPP together with an appropriately parameterized intensity function:

- *Marked Targets.* The targets are "marked" by measurements. This is a natural interpretation of the measurement process. Because of the well known Marking Theorem of PPPs (see the Appendix), the marked target PPP is equivalent to a joint PPP on the Cartesian product of the measurement and target spaces.
- *Superposition.* Individual targets are modeled as independent PPPs. Superposition is appropriate in multitarget tracking applications, so these target-specific PPPs are superposed with a known clutter PPP to obtain the aggregate PPP target model. Linear-Gaussian target motion and measurement processes are adopted.

The likelihood function of the MMIF corresponds to the superposition of the joint measurement-target PPPs, a modeling strategy that is apparently different from what has been proposed previously. The aggregate intensity function model is thus an affine, joint, linear-Gaussian sum.

Measured data are modeled as measurements for which the corresponding target states are, literally, missing. This concrete physical model of the missing data is accommodated by the Expectation-Maximization (EM) method. The result is the MMIF recursion presented in this paper.

The paper is organized as follows. The distinctive features of the MMIF are discussed in Section 2. Section 3 discusses the target and measurement model, while Section 4 discusses the joint measurement-target intensity function. Section 5 gives the likelihood function and outlines the essential details of the E- and M-steps of the EM algorithm. Section 6 explicitly defines the MMIF recursion. Concluding remarks are given in Section 7.

2 Important Features of MMIF

Heterogeneous target motion models and measurement pdfs (probability density functions) are easily incorporated into the MMIF in such a way that estimates of the multi-target state are evaluated directly by the algorithm. This sidesteps the need of a separate post-processing algorithm of the kind required by the PHD and intensity filters to extract target state (and covariance matrices) from the intensity function.

Direct estimate of the multi-target state is achieved at the cost of using a parameterized sum model for the intensity function. This strongly affects their ability to model the number of targets. However, in practice, various methods can compensate for this limitation. These methods are not discussed here.

An important feature of the MMIF is that it conforms to the "at most one measurement per target rule" but only *in the mean*, or on average. The expected number of measurements that a single target produces on the joint measurement-target space is the integral over the joint measurement-target space. The rule holds because the marked single target PPP intensity function integrates appropriately (see Eqn. (9) below).

Another important feature of the MMIF is that the EM weights depend on the Kalman filter innovations. The weights in other Gaussian sum filters often involve scaled multiples of the measurement variances, resulting in filters that are somewhat akin to nearest neighbor tracking filters. Another consequence is that the covariance matrices produced by the MMIF may be more consistent than in other EM applications to tracking, e.g., PMHT (Probabilistic Multi-Hypothesis Tracking); however, covariance consistency is not studied in this paper.

If data assignments are modeled as *independent* random variables, and these random variables are treated as missing data in the sense of EM, the resulting algorithm is little different from a PMHT-style filter that is modified so that the mixing proportions no longer sum to one. The MMIF is very different from this model because the marked PPP enforces the "at most one measurement per target rule" if only in the mean. This is a direct consequence of the EM weights including the Kalman innovations.

3 Target and Measurement Models

The MMIF uses a linear Gaussian target motion and measurement model for each target and superposes them against a known background clutter model. The target models need *not* be the same for all targets. Similarly, different sensor measurement models are also accommodated. Superposition leads to an affine Gaussian sum intensity function on the joint measurement-target space. The affine term corresponds to a specified (known) clutter PPP. (The MMIF is unrelated to the Gaussian sum methods used to approximate PHD filters.)

To reiterate and establish notation, the MMIF builds on the basic idea that a target in state x is marked with a measurement z. The multitarget state is modeled as a PPP, so the joint measurement-target vector (z, x) is a point in a realization of a PPP on the Cartesian product of the measurement space $\mathbb{R}^{n_z}$ and the target state space $\mathbb{R}^{n_x}$. As seen from the Appendix, measurement-marked target PPPs are equivalent to ordinary PPPs on the Cartesian product of the measurement and target spaces, $\mathbb{R}^{n_z} \times \mathbb{R}^{n_x}$. These joint PPPs are superposed, and the target states estimated via the EM method. The MMIF satisfies the "at most one measurement per target rule" in the mean.

Target Modeling

The multiple target state of L targets at time t_k is the vector

$$x_k \equiv (x_k(1), \ldots, x_k(L)) \,, \tag{1}$$

where $x_k(\ell) \in \mathbb{R}^{n_x}$ for $1 \leq \ell \leq L$. All targets move according to a linear Gauss-Markov model. Target states are estimated at the discrete times $t_0 < t_1 < t_2 < \ldots$, where t_0 corresponds to a starting time at which the *a priori* target pdfs are specified. The pdf of a target in state $x_{k-1}(\ell)$ at time t_{k-1} transitioning to state $x_k(\ell)$ at time t_k is

$$\begin{aligned}\Psi_{k-1}\left(x_k(\ell) \,|\, x_{k-1}(\ell)\right) \\ = \mathcal{N}\left(x_k(\ell)\,;F_{k-1}(\ell)\,x_{k-1}(\ell), Q_{k-1}(\ell)\right),\end{aligned} \tag{2}$$

where the system matrix $F_{k-1}(\ell) \in \mathbb{R}^{n_x \times n_x}$ and the process noise covariance matrix $Q_{k-1}(\ell) \in \mathbb{R}^{n_x \times n_x}$ are specified. The target motion model is equivalent to $x_k(\ell) = F_{k-1}(\ell)\,x_{k-1}(\ell) + u_{k-1}(\ell)$, where the process noise $u_{k-1}(\ell) \in \mathbb{R}^{n_x}$ is zero mean Gaussian distributed with covariance matrix $Q_{k-1}(\ell)$. The process noises are assumed independent from target to target.

Each target is modeled as a PPP. It is assumed, recursively, that the intensity function of target ℓ at time t_{k-1} is

$$f^{\ell}_{k-1|k-1}(x) = \widehat{I}_{k-1|k-1}(\ell)\,\mathcal{N}\left(x\,;\widehat{x}_{k-1|k-1}(\ell), P_{k-1|k-1}(\ell)\right), \tag{3}$$

where the MAP estimate $\hat{x}_{k-1|k-1}(\ell)$, its covariance matrix $P_{k-1|k-1}(\ell)$, and intensity $\widehat{I}_{k-1|k-1}(\ell)$ are known.

The physical interpretation of the model (3) is of considerable interest. First suppose that $\widehat{I}_{k-1|k-1}(\ell) = 1$. In this case the expression (3) would seem to say that the posterior pdf of the target state at time t_{k-1} is Gaussian with mean vector and covariance matrix of the traditional Kalman filter:

$$f^{\ell}_{k-1|k-1}(x) = \widehat{x}_{k-1|k-1}(\ell) + n_{k-1}(\ell),$$

where $n_{k-1}(\ell)$ is independent, additive, zero mean noise with covariance matrix $P_{k-1|k-1}(\ell)$. However, by assumption, $f^{\ell}_{k-1|k-1}(x)$ is the intensity function of a PPP, so this interpretation is simplistic. Instead, (3) says that the *expected* number of targets in a realization of the PPP is one. In other words, the physical interpretation of (3) is that of an ensemble average. The points in a realization of the PPP are "microtargets" (a name adapted from the microstates of thermodynamics). No matter how many microtargets are in a given realization of the PPP, they appear with independent additive noise term $n_{k-1}(\ell)$.

In the general case when $\widehat{I}_{k-1|k-1}(\ell) \neq 1$, (3) says that the expected number of microtargets is $\widehat{I}_{k-1|k-1}(\ell)$. In other words, the intensity coefficient $\widehat{I}_{k-1|k-1}(\ell)$ is a target strength parameter.

Under the target motion model (2), the predicted detected target intensity function at time t_k is

$$f^{\ell}_{k|k-1}(x) = P^D_k(\ell)\, I_k(\ell)\, \mathcal{N}\left(x\,;\widehat{x}_{k|k-1}(\ell), P_{k|k-1}(\ell)\right), \qquad (4)$$

where $P^D_k(\ell)$ is the probability of detecting target ℓ at time t_k and is assumed independent of target state x. The predicted state and covariance matrix of target ℓ are

$$\widehat{x}_{k|k-1}(\ell) = F_{k-1}(\ell)\widehat{x}_{k-1|k-1}(\ell) \qquad (5)$$

$$P_{k|k-1}(\ell) = F_{k-1}(\ell)\, P_{k-1|k-1}(\ell)\, F^T_{k-1}(\ell) + Q_{k-1}(\ell). \qquad (6)$$

The predicted intensity takes this form because the target motion model is applied to every microtarget in the PPP realizations of target ℓ. (See [2] for details.) The coefficient $I_k(\ell)$ is estimated from data at time t_k as part of the MMIF recursion.

Measurement Modeling

An arbitrary measurement $z \in \mathbb{R}^{n_z}$ at time t_k originates either from one of the L targets or from the background clutter. If it originates from the ℓ-th target with state x, then the pdf of z conditioned on x is

$$p_{Z|X_k(\ell)}(z|x) = \mathcal{N}\left(z\,; H_k(\ell)x, R_k(\ell)\right), \qquad (7)$$

where the measurement matrix $H_k(\ell) \in \mathbb{R}^{n_z \times n_x}$ and the measurement noise covariance matrix $R_k(\ell) \in \mathbb{R}^{n_z \times n_z}$ are both specified. The measurement model is equivalent to $z = H_k(\ell)x + v_k(\ell)$, where the measurement noise $v_k(\ell) \in \mathbb{R}^{n_z}$ is zero mean Gaussian distributed with covariance matrix $R_k(\ell)$. The measurement and target process noises are assumed independent.

4 Joint Measurement-Target Intensity Function of the MMIF

Measurements are modeled as marks that are associated with targets that are realizations of a target PPP. As seen from the Marking Theorem of the Appendix, a measurement-marked target PPP is equivalent to a PPP on the Cartesian product of the measurement and target spaces, that is, on $\mathbb{R}^{n_z} \times \mathbb{R}^{n_x}$. It is *not* necessary to assume that the measurement process is a PPP.

The intensity function of the joint measurement-target PPP of target ℓ in state $x \in \mathbb{R}^{n_x}$ at time t_k is, from the expression (33),

$$\lambda^{\ell}_{k|k}(z, x) = f^{\ell}_{k|k-1}(x)\, p_{Z|X_k(\ell)}(z|x). \qquad (8)$$

From the basic property of PPPs, the expected number of marked detected targets, that is, the number of targets with a measurement, is the multiple integral over $\mathbb{R}^{n_z} \times \mathbb{R}^{n_x}$:

$$\int_{\mathbb{R}^{n_z}\times\mathbb{R}^{n_x}} \lambda^{\ell}_{k|k}(z, x)\, \mathrm{d}z\, \mathrm{d}x = \int_{\mathbb{R}^{n_x}} f^{\ell}_{k|k-1}(x)\, \mathrm{d}x = P^D_k(\ell)\, I_k(\ell). \qquad (9)$$

The expected number of points (i.e., microtargets) in a realization of target model ℓ is $I_k(\ell)$, so (9) is equivalent to the "at most one measurement per target rule," in the mean. The rule applies only to detected targets, hence the factor $P^D_k(\ell)$ in (9).

Substituting (4) and (7) gives the joint measurement-target intensity function

$$\begin{aligned}\lambda^{\ell}_{k|k}(z, x) &= P^D_k(\ell)\, I_k(\ell)\, \mathcal{N}\left(x\,;\widehat{x}_{k|k-1}(\ell), P_{k|k-1}(\ell)\right) \\ &\quad \times \mathcal{N}\left(z\,; H_k(\ell)\, x_k(\ell), R_k(\ell)\right) \\ &= P^D_k(\ell)\, I_k(\ell)\, \mathcal{N}\left(x\,;\widehat{x}_{k|k}(z\,;\ell), P_{k|k}(\ell)\right) \\ &\quad \times \mathcal{N}\left(z\,;\widehat{z}_{k|k-1}(\ell), S_{k|k}(\ell)\right),\end{aligned} \qquad (10)$$

where, using $\widehat{x}_{k|k-1}(\ell)$ and $P_{k|k-1}(\ell)$ above, the usual Kalman filter equations give

Measurement covariance matrix:

$$S_{k|k}(\ell) = R_k(\ell) + H_k(\ell)\, P_{k|k-1}(\ell)\, H^T_{k-1}$$

Predicted measurement:

$$\widehat{z}_{k|k-1}(\ell) = H_k(\ell)\widehat{x}_{k|k-1}(\ell)$$

Kalman gain:

$$\begin{aligned}W_k(\ell) &= P_{k|k-1}(\ell)\, H^T_k(\ell) \\ &\quad \times \left\{H_k(\ell)\, P_{k|k-1}(\ell)\, H^T_{k-1}(\ell) + R_k(\ell)\right\}^{-1}\end{aligned}$$

Updated state covariance matrix:

$$P_{k|k}(\ell) = P_{k|k-1}(\ell) - W_k(\ell)\, H_k(\ell)\, P_{k|k-1}(\ell)$$

Information updated state estimate:

$$\widehat{x}_{k|k}(z\,;\ell) = F_{k-1}(\ell)\widehat{x}_{k-1|k-1}(\ell) + W_k(\ell)\left\{z - \widehat{z}_{k|k-1}(\ell)\right\}. \qquad (11)$$

The joint measurement-target PPPs are independent because measurements are independent when conditioned on target state, and because targets are assumed independent. The measurement clutter intensity function is

$$\lambda^0_{k|k}(z) = I_k(0)\, q_k(z), \qquad (12)$$

where $q_k(z)$ is a specified clutter pdf, i.e.,

$$\int_{\mathbb{R}^{n_z}} q_k(z)\, \mathrm{d}z = 1.$$

In the language of [4], the clutter model is a compound PPP. To ease the notational burden later in the EM method, let $\lambda^0_{k|k}(z) \equiv \lambda^0_{k|k}(z, \varnothing)$.

The joint measurement-multitarget PPP at time t_k is the superposition of target and clutter intensity functions:

$$\lambda_{k|k}(z, x) = \lambda^0_{k|k}(z) + \sum_{\ell=1}^{L} \lambda^\ell_{k|k}(z, x) \tag{13}$$

$$= I_k(0)\, q_k(z) + \sum_{\ell=1}^{L} P^D_k(\ell)\, I_k(\ell)$$

$$\times \mathcal{N}\left(x\,;\widehat{x}_{k|k}(z\,;\ell), P_{k|k}(\ell)\right) \mathcal{N}\left(z\,;\widehat{z}_{k|k-1}(\ell), S_{k|k}(\ell)\right).$$

This affine sum parameterizes the likelihood function of the MMIF filter. The EM method uses it in the next section to derive a recursion for estimating target states and the intensity coefficients.

5 Likelihood Function

The MMIF recursion for linear-Gaussian target and measurement models is derived in this section via the EM method. Background on the EM method is widely available; the book [1] is an excellent reference and guide to the literature.

Denote the number of measurements at time t_k by $m_k \geq 1$ and the measurements themselves by

$$z_k(1:m_k) = \{z_k(1), \ldots, z_k(m_k)\},$$

where $z_k(j) \in \mathbb{R}^{n_z}$, $j = 1, \ldots, m_k$. (Details for the special case $m_k = 0$ are omitted.) In a joint measurement-target PPP, every measurement z is always paired with a point x in target state space, but whether or not x corresponds to a target or to clutter is unknown. Denote the target states associated with the measurements (marks) by

$$x_k(1:m_k) = \{x_k(1), \ldots, x_k(m_k)\},$$

where $x_k(j) \in \mathbb{R}^{n_x}$, $j = 1, \ldots, m_k$. The paired data are

$$\mathcal{Z}_k = \{(z_k(j), x_k(j)) : j = 1, \ldots, m_k\}.$$

Because the target model is a PPP, the data $\mathcal{Z}_k$ are a realization of the measurement-target PPP with intensity (13). Its likelihood function is

$$p(\mathcal{Z}_k) = e^{-\int_{\mathbb{R}^{n_z}\times\mathbb{R}^{n_x}} \lambda_{k|k}(z,x)\,\mathrm{d}z\,\mathrm{d}x} \prod_{j=1}^{m_k} \lambda_{k|k}(z_k(j), x_k(j))$$

$$= e^{-I_k(0) - \sum_{\ell=1}^{L} P^D_k(\ell) I_k(\ell)}$$

$$\times \prod_{j=1}^{m_k} \left\{ \lambda^0_{k|k}(z_k(j)) + \sum_{\ell=1}^{L} \lambda^\ell_{k|k}(z_k(j), x_k(j)) \right\}, \tag{14}$$

where (9) is used in the last equation.

E-Step

The difficulty is that there are as many unknown target states as there are data, while there are L target modes and a clutter mode. Denote the states of the L target modes by

$$\chi_k(1:L) = \{\chi_k(1), \ldots, \chi_k(L)\}, \tag{15}$$

where $\chi_k(j) \in \mathbb{R}^{n_x}$, $j = 1, \ldots, m_k$. The clutter mode is mode zero, and its state is $\chi_k(0) = \varnothing$. The unobserved target state $x_k(j)$ of the measurement $z_k(j)$ corresponds to one of the L target modes or to clutter. Let σ_j denote the index of this mode, so that $\sigma_j \in \{0, 1, \ldots, L\}$. It is now assumed that

$$x_k(j) = \chi_k(\sigma_j), \quad j = 1, \ldots, m_k. \tag{16}$$

In other words, measurements that arise from the same mode have exactly the same target state. The constraints (16) violates the exact form of the "at most one measurement per target rule", but it is not violated *in the mean*. The target states to be estimated are $\chi_k(1:L)$.

The superposition in (14) is a clear indication of the utility of the EM method for computing MAP estimates. In EM parlance, (14) is the incomplete data pdf. It is natural (indeed, other choices seem contrived here) to let the indices $\sigma \equiv \{\sigma_1, \ldots, \sigma_{m_k}\}$ denote the missing data. The complete data pdf is defined by

$$p(\mathcal{Z}_k, \sigma) = e^{-I_k(0) - \sum_{\ell=1}^{L} P^D_k(\ell) I_k(\ell)} \prod_{j=1}^{m_k} \lambda^{\sigma_j}_{k|k}\left(z_k(j), \chi_k(\sigma_j)\right). \tag{17}$$

Let $I_k(0:L) \equiv (I_k(0), I_k(1), \ldots, I_k(L))$. The posterior pdf of σ is, by the definition of conditioning,

$$p\left(\sigma \,\middle|\, \chi_k(1:L), I_k(0:L)\right) = \frac{p(\mathcal{Z}_k, \sigma)}{p(\mathcal{Z}_k)}$$

$$= \prod_{j=1}^{m_k} w_{\sigma_j}(z_k(j)\,; \chi_k(1:L), I_k(0:L)), \tag{18}$$

where, for $1 \leq \ell \leq L$, the weights for an arbitrary measurement z are given by

$$w_\ell(z\,; \chi_k(1:L), I_k(0:L)) = \frac{D_\ell(z)}{D_0(z)}. \tag{19}$$

where the numerator $D_\ell(z)$ is given by

$$D_\ell(z) = P^D_k(\ell)\, I_k(\ell)$$

$$\times \mathcal{N}\left(\chi_k(\ell)\,;\widehat{x}_{k|k}(z\,;\ell), P_{k|k}(\ell)\right) \mathcal{N}\left(z\,;\widehat{z}_{k|k-1}(\ell), S_{k|k}(\ell)\right) \tag{20}$$

and the denominator $D_0(z)$ is given by

$$D_0(z) = I_k(0) q_k(z) + \sum_{\ell=1}^{L} P^D_k(\ell) I_k(\ell)$$

$$\times \mathcal{N}\left(\chi_k(\ell);\widehat{x}_{k|k}(z;\ell), P_{k|k}(\ell)\right) \mathcal{N}\left(z;\widehat{z}_{k|k-1}(\ell), S_{k|k}(\ell)\right) \tag{21}$$

The weight for $\ell = 0$ is

$$w_0(z; \chi_k(1:L), I_k(0:L)) = \frac{I_k(0)\, q_k(z)}{D_0(z)}. \qquad (22)$$

The coefficient $e^{-I_k(0) - \sum_{\ell=0}^{L} P_k^D(\ell) I_k(\ell)}$ cancels out in the weight calculation. The weights are ratios of intensities. They are the probabilities that the measurement z is generated by target ℓ, or by clutter if $\ell = 0$.

Let $r = 0, 1, \ldots$ be the EM iteration index, and let $\chi_k^{(0)}(1:L)$ and $I_k^{(0)}(0:L)$ be specified initial values of the target states and their intensity coefficients. The EM auxiliary function is the conditional expectation

$$Q\Big(\chi_k(1:L), I_k(0:L) \,\Big|\, \chi_k^{(r)}(1:L), I_k^{(r)}(0:L)\Big) = \sum_{j=1}^{m_k} \{\log p(\mathcal{Z}_k, \sigma)\}\; w_\ell\Big(z; \chi_k^{(r)}(1:L), I_k^{(r)}(0:L)\Big). \qquad (23)$$

Proceeding algebraically in the manner typical of applications of EM methods, and dropping terms that do not depend on $\chi_k(1:L)$ and $I_k(0:L)$ gives the simplified expression

$$\begin{aligned} Q\Big(\chi_k(1:L), I_k(0:L) \Big| \chi_k^{(r)}(1:L), I_k^{(r)}(0:L)\Big) &= -I_k(0) - \sum_{\ell=1}^{L} P_k^D(\ell) I_k(\ell) \\ &+ \sum_{\ell=0}^{L} \sum_{j=1}^{m_k} w_\ell\Big(z; \chi_k^{(r)}(1:L), I_k^{(r)}(0:L)\Big) \log \lambda_{k|k}^{\ell}(z_k(j), \chi_k(\ell)). \end{aligned} \qquad (24)$$

M-Step

Maximizing the auxiliary function with respect to $\chi_k(1:L)$ and $I_k(0:L)$ gives the EM recursion. As is evident from Eqn. (24), the auxiliary function separates into a sum over the L target models. Consequently, maximizing Q over all targets is equivalent to maximizing Q over each target model separately. This requires taking the gradient of Q with respect to $I_k(\ell)$ and $\chi_k(\ell)$, and solving for the updates. Details are given in the next section.

6 MMIF Recursion

The EM update for the intensity coefficient of the ℓ-th target is given by

$$I_k^{(r+1)}(\ell) = \frac{1}{P_k^D(\ell)} \sum_{j=1}^{m_k} w_\ell\Big(z_k(j); \chi_k^{(r)}(1:L), I_k^{(r)}(0:L)\Big). \qquad (25)$$

The factor $P_k^D(\ell)$ cancels the same factor in the weights (19). For clutter, $\ell = 0$, the updated intensity coefficient is

$$I_k^{(r+1)}(0) = \sum_{j=1}^{m_k} w_0\Big(z_k(j); \chi_k^{(r)}(1:L), I_k^{(r)}(0:L)\Big). \qquad (26)$$

These updates accord well with the interpretation of the weights.

Finding the updated state for target ℓ is little different. Setting the gradient with respect to $\chi_k(\ell)$ equal to zero and solving gives the update

$$\chi_k^{(r+1)}(\ell) = \frac{\sum_{j=1}^{m_k} w_\ell\Big(z_k(\ell); \chi_k^{(r)}(1:L), I_k^{(r)}(0:L)\Big)\, \widehat{x}_{k|k}(z_k(j); \ell)}{\sum_{j=1}^{m_k} w_\ell\Big(z_k(\ell); \chi_k^{(r)}(1:L), I_k^{(r)}(0:L)\Big)}. \qquad (27)$$

A more intuitive way to write the result is to substitute for $\widehat{x}_{k|k}(z_k(j); \ell)$ using (11). By linearity, the updated state is given by the Kalman filter

$$\begin{aligned} \chi_k^{(r+1)}(\ell) &= F_{k-1}(\ell)\widehat{x}_{k-1|k-1}(\ell) \\ &\quad + W_k(\ell)\left\{\widetilde{z}_{k|k}^{(r+1)}(\ell) - \widehat{z}_{k|k-1}(\ell)\right\}, \end{aligned} \qquad (28)$$

where the "synthetic" measurement for target ℓ is defined by

$$\widetilde{z}_{k|k}^{(r+1)}(\ell) = \frac{\sum_{j=1}^{m_k} w_\ell\Big(z_k(\ell); \chi_k^{(r)}(1:L), I_k^{(r)}(0:L)\Big)\, z_k(j)}{\sum_{j=1}^{m_k} w_\ell\Big(z_k(\ell); \chi_k^{(r)}(1:L), I_k^{(r)}(0:L)\Big)}. \qquad (29)$$

This concludes one iteration of the EM algorithm. (The iteration can be rewritten explicitly in terms of innovations. For $L = 1$, the innovation form resembles the probabilistic data association filter [5].)

On convergence, at say iteration r_{last}, the MAP estimates of the state of target ℓ and its intensity are

$$\widehat{I_{k|k}}(\ell) = I_{k|k}^{(r_{last})}(\ell), \qquad 0 \le \ell \le L, \qquad (30)$$

$$\widehat{x}_{k|k}(\ell) = \chi_{k|k}^{(r_{last})}(\ell), \qquad 1 \le \ell \le L. \qquad (31)$$

EM iteration stopping criteria are discussed elsewhere.

The M-step of the EM update recursions are explicitly solvable because the surveillance region is the entire measurement space $\mathbb{R}^{n_z}$. Bounded surveillance regions, or regions of regard, are necessary in practice. As mentioned in [4] (see also [8, Appendix B]), the update equations must be appropriately modified for bounded regions. If all targets are well inside the surveillance region, the equations here are excellent approximations.

7 Concluding Remarks

The MMIF algorithm is derived using a marked PPP multitarget intensity function. The marks correspond to measurements. The marked PPP is equivalent to an ordinary PPP on the Cartesian product of the measurement and target state spaces. The intensity function of this joint PPP is parameterized using an affine linear-Gaussian sum, where the affine term is a specified clutter PPP and the Gaussian pdfs correspond to measurement-target pairs.

The method of EM is then applied to develop the MMIF algorithm for multiple target states and their intensities. The Kalman innovations enter the EM weights in a natural manner, an aspect of the MMIF model that suggests good performance in practice. The "at most one measurement per target" rule holds in the mean, primarily because the EM weights depend on the innovations.

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Appendix. Joint Measurement-Target Poisson Point Processes and the Marking Theorem

Marked PPPs are defined in the first section below. In the next section, marked PPPs are shown to be PPPs on a Cartesian product space.

Marked Poisson Point Processes

Marked PPPs model problems in which the points are accompanied by a "mark." The mark is commonly called a "feature vector" in applications. The mark can be discrete, continuous, or discrete-continuous. An old example is in forestry, where a point represents the location of a tree, and its mark comprises one or more of the following kinds of information: its species and health (discrete/categorical), circumference at a fixed height above the ground (continuous), or both. Other examples are easily conceived.

Realizations of a marked PPP are in principle almost as easy to generate as those of a PPP:

- Use the standard two-step procedure to generate a realization
$$\xi = (m, \{x_1, \ldots, x_m\})$$
of a PPP Ξ with intensity $\lambda(s)$ on the space $\mathcal{S}$. Given m, the points x_j are i.i.d. samples of the random variable X whose pdf is $p_X(x) = \lambda(x) / \int_{\mathcal{S}} \lambda(s)\, ds$.
- The mark is a random variable U on a mark space $\mathcal{U}$ with conditional pdf $p_{U|X}(u\,|\,x)$. Given the PPP realization ξ, generate the marks
$$(u_1, \ldots, u_m) \subset \mathcal{U}$$
as independent realizations of $p_{U|X}(\cdot\,|\,x_j)$, $j = 1, \ldots, m$.
- Pair the points with their corresponding marks:
$$\xi' \equiv (m, \{(x_1, u_1), \ldots, (x_m, u_m)\}) . \qquad (32)$$

The realization of the marked PPP is ξ'.

The mark space $\mathcal{U}$ can be very general, but it is typically either a discrete set or a subset of the Euclidean space $\mathbb{R}^\kappa$. For the application in this paper, the targets are the points of a PPP, the measurements are the marks, and the measurement likelihood function is the conditional mark pdf.

In the simplest case, the marks are independent of ξ, so that
$$p_{U|X}(u\,|\,x_j) \equiv p_U(u) .$$
The marks in this case are independent of the locations of the points in ξ as well as the number m. This kind of marked PPP is called a compound PPP by Snyder, and its theory is well developed in [3, Chap. 3] and [4, Chap. 4].

For EM estimation of Gaussian sums, marks are introduced as part of the complete data; that is, the marks are the missing data of the EM method. In this application, the mark space comprises the indices of the components of the sum, so that $\mathcal{U} = \{1, \ldots, L\}$ when there are L components.

Yet another example, one that could have been discussed as a marked PPP but was not, is the intensity filter of [6] and [8, Chapter 6] when the target PPP is split during the prediction step into the detected and undetected target PPPs. In this case, the detection process is equivalent to a marking procedure with marks $\mathcal{U} = \{0, 1\}$, where zero/one denotes target non-detection/detection. The multisensor intensity filter [7] can also be treated as a marked PPP.

Marking Theorem

Marked PPPs are intuitive models with a rich structure that enables them to model phenomena in diverse applications. The mark structure might make them seem fundamentally different from ordinary PPPs, but this is not so. Marked PPPs are equivalent to PPPs on the Cartesian product of the space $\mathcal{S}$ and the mark space $\mathcal{U}$ with a joint intensity function μ given by

$$\mu(x, u) = p_{U|X}(u\,|\,x)\,\lambda(x)\,. \tag{33}$$

This result is important as well as insightful. The similarity of (33) to conditional factorization is self evident.

First observe that the realization ξ' of (32) is an element of the PPP event space $\mathcal{E}(\mathcal{S}\times\mathcal{U})$, so it has the form of a realization of a PPP on $\mathcal{S}\times\mathcal{U}$. Since the two-step procedure is the definition of a PPP, it is only necessary to verify that the intensity function of the point process with realizations ξ' takes the form (33).

The characteristic function of a finite point process has form required by Campbell's Theorem if and only if the finite point process is a PPP. A proof is given in [2]. In light of this result, it is enough to show that for functions $f(x, u)$ defined on the Cartesian product $\mathcal{S}\times\mathcal{U}$ the characteristic function of the random sum

$$\begin{aligned} F(\xi') &\equiv F(m, (x_1, u_1), \ldots, (x_m, u_m)) \\ &= \sum_{j=1}^{m} f(x_j, u_j) \end{aligned}$$

is in the form given by Campbell's Theorem, namely,

$$\begin{aligned} E\left[e^{-F}\right] &= \int_{\mathcal{S}}\int_{\mathcal{U}} \left(e^{-f(x,u)} - 1\right) p_{U|X}(u\,|\,x)\,\lambda(x)\,\mathrm{d}x\mathrm{d}u \\ &= \int_{\mathcal{S}\times\mathcal{U}} \left(e^{-f(x,u)} - 1\right) \mu(x, u)\,\mathrm{d}x\mathrm{d}u\,. \end{aligned} \tag{34}$$

The random variables $f(X_j, U_j)$ are independent given m, so the expectation of $F(\xi')$ with respect to the marks, conditioned on the points $x_1, \ldots, x_m$, is

$$\begin{aligned} E\left[e^{-F}\,\middle|\,x_1, \ldots, x_m\right] &= E_{U_1\cdots U_m|X_1\cdots X_m}\left[e^{-\sum_{j=1}^{m} f(x_j, U_j)}\right] \\ &= E_{U_1\cdots U_m|X_1\cdots X_m}\left[\prod_{j=1}^{m} e^{-f(x_j, U_j)}\right] \\ &= \prod_{j=1}^{m} E_{U_j|X_j}\left[e^{-f(x_j, U_j)}\right] \\ &= \prod_{j=1}^{m} \int_{\mathcal{U}} e^{-f(x_j, u_j)}\,p_{U|X}(u_j\,|\,x_j)\,\mathrm{d}u_j \\ &= e^{-\sum_{j=1}^{m} g(x_j)}\,, \end{aligned} \tag{35}$$

where, for any $x \in \mathcal{S}$,

$$g(x) = -\log \int_{\mathcal{U}} e^{-f(x,u)}\,p_{U|X}(u\,|\,x)\,\mathrm{d}u\,. \tag{36}$$

Applying Campbell's Theorem gives the expectation of the expression (35) with respect to the PPP with intensity function $\lambda(x)$:

$$\begin{aligned} E\left[e^{-F}\right] &= E_{\Xi}\left[e^{-\sum_{j=1}^{M} g(X_j)}\right] \\ &= \exp\left\{\int_{\mathcal{S}} \left(e^{-g(x)} - 1\right) \lambda(x)\,\mathrm{d}x\right\}. \end{aligned}$$

Substituting (36) gives

$$\begin{aligned} &E\left[e^{-F}\right] \\ &= \exp\left\{\int_{\mathcal{S}} \left(\int_{\mathcal{U}} e^{-f(x,u)}\,p_{U|X}(u\,|\,x)\,\mathrm{d}u - 1\right) \lambda(x)\,\mathrm{d}x\right\} \\ &= \exp\left\{\int_{\mathcal{S}} \left(\int_{\mathcal{U}} \left(e^{-f(x,u)} - 1\right) p_{U|X}(u\,|\,x)\,\mathrm{d}u\right) \lambda(x)\,\mathrm{d}x\right\}. \end{aligned}$$

The last expression is equivalent to (34).

This result applies with only minor changes when the mark space is discrete. For further discussion, see the excellent book by Kingman [2].